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Quillen Stratification for Modules

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Introduction

Let G be a finite group and k a fixed algebraically closed field of characteristic $p>0$. If p is odd, let H_G be the subring of $H^*(G,k)$ consisting of elements of even degree; following [20–22] we take $H_G=H^*(G,k)$ if $p=2$, though one could just as well use the subring of elements of even degree for all p. H_G is a finitely generated commutative k-algebra [13], and we let V_G denote its associated affine variety $\operatorname{Max} H_G$. If M is any finitely generated kG-module, then the *cohomology variety* $V_G(M)$ of M may be defined as the support in V_G of the H_G-module $H^*(G,M)$ if G is a p-group, and in general as the largest support of $H^*(G, L\otimes M)$, where L is any kG-module [4, 9]. A module L with each irreducible kG-module as a direct summand will serve.

D. Quillen [20–22] proved a number of beautiful results relating V_G to the varieties V_E associated with the various elementary abelian p-subgroups E of G, culminating in his stratification theorem [20, 22]. This theorem gives a piecewise description of V_G almost explicitly in terms of the subgroups E and their normalizers in G. A well-known corollary is that $\dim V_G=\max_E \dim V_E$, where E ranges over the elementary abelian p-subgroups of G (or, as it is usually stated, the Krull dimension of H_G is the maximum of the ranks of these elementary abelian subgroups).

For some time after Quillen's work no further progress was made. Then Chouinard [10] proved the first related result for cohomology with nontrivial coefficients. We give below a survey of the results that have been obtained since then before describing our own results.

Quillen's original proof of the stratification theorem for compact Lie groups [21, 22] used equivariant cohomology; he also gave an algebraic treatment for finite groups [20], but still needed equivariant cohomology for one key lemma. An algebraic proof of this lemma was then given by Quillen

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and Venkov [23], using an elegant theorem of Serre [24]. Using these later methods, Chouinard proved that a kG-module is projective if and only if its restriction to every elementary abelian p-subgroup is projective. Although not explicitly a generalization of Quillen's results, Chouinard's theorem can be viewed as providing a stratification theorem for $V_G(M)$ in the case where $V_G(M)$ is the single point 0.

Alperin realized that Chouinard's result could be interpreted in terms of his own notion of the "complexity" of a module, defined in terms of the rate of growth of a minimal projective resolution. From this point of view, Chouinard's theorem said that the property of having complexity 0 was determined on the elementary abelian p-subgroups. Together with Evens, Alperin [2] proved that the same was true for complexity n, with n any nonnegative integer. They also showed that the case of the trivial module (which has highest possible complexity) implied Quillen's corollary on the Krull dimension.

Meanwhile Jon Carlson realized that the Chouinard and Alperin-Evens results fit nicely with some of his own ideas, which were partly inspired by a lemma of Dade [11]. This lemma said that nonprojective kE-modules, for E elementary, could be detected on certain well-behaved subalgebras of kE. These subalgebras are isomorphic to the group algebra of a cyclic group of order p, though not necessarily associated with any subgroup of E. Carlson was able to reprove and slightly strengthen the Alperin-Evens complexity results [8], and introduced the notion of the "rank variety," which we denote here by $V_E^r(M)$. This variety is completely determined by elementary considerations, without any cohomology (cf. § 1). Nevertheless, Carlson [7, 9] was able to show at least that $V_E^r(M) \subseteq V_E(M)$, with equality of dimensions. (The latter fact was actually proved somewhat earlier by Alperin's student Ove Kroll [18].) Carlson conjectured that $V_E^r(M) = V_E(M)$.

In another development, Avrunin [4] redid the Alperin-Evens results from the standpoint of commutative algebra, focusing attention on the rings involved rather than their dimensions, and raised the question of generalizing Quillen's stratification theorem to modules. A similar investigation, formulated in terms of varieties, was undertaken at about the same time by Alperin and Evens [3] in response to a question of Serre and a suggestion of Scott. They proved that there is a surjection $\coprod V_E(M) \to V_G(M)$, where E ranges over the elementary subgroups of G. (This also follows from Avrunin's formulation.) Each $V_E(M) \to V_G(M)$ is a finite morphism, and the dimension of $V_G(M)$ may be interpreted as the complexity of M, so this result also may be considered as a refinement of the original Alperin-Evens complexity theorem.

Avrunin and Scott now concentrated on generalizing Quillen's stratification theorem to modules. The surjection above is not quite sufficient; what is really needed is the statement that any point in $V_G(M)$ in the image of V_E is in fact in the image of $V_E(M)$. This difficulty persisted, and was discussed at length by Scott, Alperin, and Carlson at the 1980 Oberwolfach conference on integral representation theory. Not long afterward, Alperin [1] was able to show that the required fact could be obtained as a consequence of Carlson's conjecture, if the latter were true. His reduction made use of a new "tensor product theorem" Carlson presented at the conference, giving $V_E^r(M \otimes_k N) = V_E^r(M) \cap V_E^r(N)$ for the rank variety in the elementary case.

We give here a proof of the stratification theorem (3.2), obtained by proving Carlson's conjecture (1.1). The approach was motivated by Alperin's reduction, though in our version the tensor product theorem emerges at the end, this time formulated for arbitrary G (3.5). We would like to express our gratitude to Jon Alperin and Jon Carlson for their roles in this research, and we thank Peter Donovan for some earlier conversations.

We are also able generalize some of Quillen's other results to the module case; in particular we obtain a "glueing theorem" (3.4) for the varieties $V_E(M)$, describing $V_G(M)$ as an inductive limit "up to inseparable isogeny." It is worth pointing out that results such as this largely describe $V_G(M)$ independently of any cohomology, because of the validity of Carlson's conjecture. Nevertheless, the reader should be cautioned that they do not give a complete answer, in the sense of exactly specifying the coordinate ring, just as Quillen's original results for the case $M = k$ do not completely describe the coordinate ring of V_G (not to mention the possibly non-reduced ring H_G). The powers of p that arise (cf. 2.3, 2.4, 3.3, 3.5) should be taken seriously. However, if one regards V_G as known, then the stratification theory together with the rank varieties already gives $V_G(M)$ without further reference to cohomology.

Finally we mention that other items of interest in this paper include a reformulation of Carlson's conjecture in terms of restricted Lie algebras (1.1), and a discussion of stratification in terms of vertices and sources (2.1). In the process of developing our module stratification theory we have found it convenient to give an exposition of some of Quillen's results. In particular, we at least state all the theorems we use and hope our account may serve as a readable introduction for the reader unfamiliar with Quillen's work.

Our main results in this paper were announced in [5].

§ 1. Carlson's Conjecture

Let E be an elementary abelian p-group, and choose a k-subspace L of kE with $J = L \oplus J^2$, where J is the kernel of the augmentation $kE \to k$. For example, take $L = \coprod_{i=1}^{n} k(e_i - 1)$, where $\{e_1, \ldots, e_n\}$ is an $\mathbb{F}_p$-basis of E. Clearly, L generates kE as a k-algebra and $\ell^p = 0$ for each $\ell \in L$. It follows that if $\{\ell_1, \ldots, \ell_n\}$ is a basis of L, then the monomials $\ell_1^{s_1} \ell_2^{s_2} \ldots \ell_n^{s_n}$ with $0 \leqq s_i < p$ for each i span kE and, by dimension considerations, form a k-basis.

Thus the group $\langle 1 + \ell_1, \ldots, 1 + \ell_n \rangle$ is a conjugate to E under an algebra automorphism of kE, and Carlson consequently called this group and its subgroups *shifted* subgroups of E [9] (cf. also [6] and [7]). For many purposes, including the restriction and induction of representations these shifted subgroups can be treated as if they were in fact subgroups.

With this terminology, Carlson defined, for any finitely generated kE-module $M \neq 0$, the rank variety $V_L^r(M)$ as 0 together with all nonzero $\ell \in L$ for which $M|_{\langle 1+\ell \rangle}$ is not free. (For $M = 0$, the correct definition is $V_L^r(M) = \emptyset$.) His definition is motivated by a lemma of Dade [11] to the effect that, if M is not free, then $M|_{\langle 1+\ell\rangle}$ is not free for some nonzero $\ell\in L$. Carlson showed that $V_L^r(M)$ is a variety with dimension equal to that of $V_E(M)$, exhibited an injection $V_L^r(M)\to V_E(M)$, proved that his map is an isomorphism if and only if $V_E(M)$ satisfies the tensor product theorem disussed in the introduction, and conjectured that $V_L^r(M)\cong V_E(M)$.

We prefer to reformulate this conjecture and some of the ideas above in terms of restricted Lie algebras. This reformulation seems to clarify some of the issues, and was of at least psychological value in finding a proof. However, the reader who prefers shifted subgroups should have no difficulty rewriting the proofs to suit his tastes. Still another alternative for the reader more comfortable with group cohomology would be to give proofs in terms of infinitesimal group schemes [12].

From the first paragraph of this section it is clear that kE is the restricted enveloping algebra $u(L)$ of L, regarded as a commutative restricted Lie algebra with trivial p-th power. One defines H_L, V_L, and $V_L(M)$ as in the group case, and, since restricted L-modules are just modules for $u(L)$, one sees easily that $H_L=H_E$, $V_L=V_E$, and $V_L(M)=V_E(M)$. (The Hopf algebra structures on kE $=u(L)$ are different in its two roles, but this does not affect the relevant cohomology products [19, VIII Prop. 4.5].) The set $V_L^r(M)\subseteq L$ is now defined as the union of all 1-dimensional k-subspaces S of L (automatically restricted Lie subalgebras) for which $M|_S$ is not projective. Clearly this agrees with the previous definition.

A technical point now arises, in that we need a natural "Bockstein" homomorphism $\beta\colon H^1(L,K)\to H^2(L,K)$. This is obtained by defining, for any 1-cocycle $f\in Z^1(u(L),k)$, a 2-cocycle $\beta(f)\in Z^2(u(L),k)$ by the rule

$$\beta(f)(x,y)=\frac{(f(x))+f((y))^p-f(x)^p-f(y)^p}{p}\quad x,y\in u(L).$$

Passing to cohomology gives the desired natural homomorphism. For L a 1-dimensional trivial restricted Lie algebra (i.e., commutative with zero p-th power), one sees easily that β is a cohomology isomorphism, and the effect of β for a general trivial algebra L can be analyzed by decomposing L as a product of 1-dimensional algebras. In particular, β is injective, and for p odd the subalgebra $\tilde H_L$ of H_L generated by $\beta H^1(L,k)$ is a symmetric algebra and complements the radical of H_L. Since $H^1(L,k)\cong L^*$, this gives a natural identification $L\cong\operatorname{Max}\tilde H_L\cong\operatorname{Max}H_L=V_L$. Similar remarks apply for $p=2$, since H_L is a polynomial algebra in $H^1(L,k)$; here we write $\tilde H_L=H_L$.

For a subalgebra T of L, let $t_{L,T}\colon V_T\to V_L$ be the map induced by restriction $H_L\to H_T$. The naturality of our construction shows that *in the identification* $L\cong V_L$, *we have* $T\cong t_{L,T}V_T$ *for any subalgebra* T *of* L. Next one proves exactly as in [4] that $t_{L,T}V_T(M)\subseteq V_L(M)$. (The main ingredient in this is the identity $M|_T|^L\cong M\otimes 1_T|^L$, where 1_T is the trivial 1-dimensional T-module. The tensor product is taken in the sense of Lie algebras, or, more generally, for any Hopf algebra structure on $u(L)$ in which $u(T)$ is a Hopf subalgebra.) If S is a 1-dimensional subalgebra of L and $M|_S$ is not projective, then $V_S(M)=V_S$, giving

$S \cong t_{L,S} V_S \subseteq V_L(M)$. Thus, if $V_L^r(M) \subseteq L$ is identified with a subset of V_L, we have $V_L^r(M) \subseteq V_L(M)$.

(1.1) **Theorem** *(Carlson's conjecture). Let L be a finite dimensional trivial restricted Lie algebra. Identifying L with V_L as above, we have $V_L^r(M) = V_L(M)$ for any finite dimensional restricted L-module M.*

Proof. It remains only to prove $V_L(M) \subseteq V_L^r(M)$. Since $H^*(L, M)$ is graded, the variety $V_L(M)$ is a union of lines. Let S be a 1-dimensional subalgebra of L with $S \cong t_{L,S} V_S \subseteq V_L(M)$. We have to show $M|_S$ is not projective.

If $M|_S$ is projective, then a spectral sequence argument gives $H^*(L/S, M^S) \xrightarrow{\sim} H^*(L, M)$, where the isomorphism is inflation followed by the map on cohomology induced by the inclusion $M^S \subseteq M$. If we regard $H^*(L, M)$ as an $H^*(L/S, k)$-module via the inflation map $H^*(L/S, k) \to H^*(L, k)$, then our map $H^*(L/S, M^S) \xrightarrow{\sim} H^*(L, M)$ is an isomorphism of $H^*(L/S, k)$-modules. It follows from Evens' theorem [13] that $H^*(L, M)$ is finitely generated as an $H^*(L/S, k)$-module, hence finitely generated as an $H_{L/S}$-module. (This could also be proved here by induction on the dimension of M, using the finite generation of the k-algebra $H^*(L/S, k)$ and the long exact sequence of cohomology).

On the other hand, the map $S \subseteq L \to L/S$ factors through 0, so the ideal $H_{L/S}^+$ consisting of all elements of positive degree in $H_{L/S}$ maps to 0 under $H_L \to H_S$. That is, $H_{L/S}^+$ (or, more precisely, its inflation) is contained in the ideal $\mathscr{P}$ of $S = t_{L,S} V_S$ in H_L, and so the H_L-module $H^*(L, M)/\mathscr{P} H^*(L, M)$ is a finite dimensional k-space. It follows from Nakayama's lemma that the support of this module is just the intersection of S and the support $V_L(M)$ of $H^*(L, M)$. Thus $V_L(M)$ contains only finitely many points of S, a contradiction which completes the proof.

The original form of Carlson's conjecture is, of course, an immediate consequence. In particular, the variety $V_L^r(M) = V_E(M)$ does not depend on the choice of L. This yields the following curious result, due originally to Carlson [9, Lemma 6.4].

(1.2) **Corollary.** *Let E be an elementary abelian p-group and let J denote the augmentation ideal as above. Suppose j and n are elements of kE with $j \in J \setminus J^2$ and $n \in J^2$. Let M be a finite dimensional kE-module, and consider the restriction of M to the multiplicative subgroups $\langle 1+j \rangle$ and $\langle 1+j+n \rangle$, generated by $1+j$ and $1+j+n$, respectively, in the group of units of kE. Then $M|_{\langle 1+j \rangle}$ is free if and only if $M|_{\langle 1+j+n \rangle}$ is free.*

The next corollary is crucial for the main result in § 3. Recall once more that a restricted Lie subalgebra of L is just a k-subspace.

(1.3) **Corollary.** *Let L be a finite dimensional trivial restricted Lie algebra, T a subalgebra of L, and M a finite dimensional restricted L-module. Then $t_{L,T}^{-1} V_L(M) = V_T(M)$. Similarly, if E is an elementary abelian p-group, $T \subseteq kE$ is a shifted subgroup of E, and M is a kE-module, then $t_{E,T}^{-1} V_E(M) = V_T(M)$ (where $t_{E,T}\colon V_T \to V_E$ is the obvious transfer map).*

Proof. We have $T \cong t_{L,T} V_T$. If S is 1-dimensional subspace of T contained in $V_L(M)$, then $M|_S = (M|_T)|_S$ is not projective and $S \subseteq V_T(M)$. The first assertion follows, and the translation in terms of shifted subgroups is an immediate consequence.

§ 2. A Review of Quillen's Results

Let G be a finite group and V_G the cohomology variety for G as before. Again we let $t_{G,H}$: $V_H \to V_G$ denote the transfer map associated with a subgroup H, induced by restriction $H_G \to H_H$. If $g \in G$, conjugation induces a map $H_{gHg^{-1}} \to H_H$ and, correspondingly, a map $V_H \to V_{gHg^{-1}}$, which we shall call *conjugation by* g. The following theorem reformulates some of Quillen's stratification theory in a manner reminiscent of Green's "vertices and sources" [14, 15]; one might call E below a "vertex" and s a "source" for x. In the remainder of the section, we show how most of the rest of the theory follows.

(2.1) **Theorem.** *Fix $x \in V_G$. Then there exists an elementary abelian p-subgroup E and an element $s \in V_E$ such that*

$$x = t_{G,E}(s).$$

Moreover, all such pairs (E, s) which also satisfy the minimality condition

$$s \neq t_{E,F}(u) \quad \text{for any } F < E \text{ and } u \in V_F$$

are conjugate under G.

Proof. This is contained in [22, Prop. 9.6], or [20, Lemma 3.7] together with the ensuing discussion.

We remark that the method of proof in [20] is very much in the spirit of Green's arguments in the reference above. However, the cohomology ring and multiplicative norms do not quite fit the formalism of [15], and rather seem to reveal a genuinely new aspect of the theory.

We now derive Quillen's stratification theorem from (2.1). For each elementary subgroup E of G, put $V_E^+ = V_E \setminus \bigcup_{F<E} t_{E,F} V_F$; the minimality condition on (E, s) in (2.2) now reads "$s \in V_E^+$". Put $V_{G,E} = t_{G,E} V_E$ and $V_{G,E}^+ = t_{G,E} V_E^+$. The map $t_{G,E}$ is a finite morphism [17], hence closed, so $V_{G,E}$ is closed in V_G. Also, the preceding theorem shows $t_{G,E}^{-1} V_{G,E}^+ = V_E^+$, so $V_{G,E}^+$ is open in $V_{G,E}$.

(2.2) **Theorem** (*Quillen's stratification theorem* [20, 22]). *The variety V_G is the disjoint union of its subvarieties $V_{G,E}^+$, where E ranges over a set of representatives for the conjugacy classes of elementary abelian p-subgroups of G. Moreover, each of the varieties $V_{G,E}^+$ and V_E^+ is affine, the group $N_G(E)/C_G(E)$ acts freely on V_E^+, and the map $t_{G,E}$ induces a bijective finite morphism*

$$V_E^+/(N_G(E)/C_G(E)) \to V_{G,E}^+.$$

Quillen uses the word "homeomorphism" in his statement of the result; a bijective finite morphism is, of course, a homeomorphism in the Zariski topology. But elsewhere he refers to the map as an "inseparable isogeny" and discusses it in terms of p-powers. This is explained by the following lemma. The result is largely already in the appendix to [22], though it plays a somewhat more central role in our treatment. We write $R(X)^q$ for $\{r^q | r \in R(X)\}$.

(2.3) **Lemma.** *Let $f: X \to Y$ be a finite bijective morphism of varieties over k. Then*

(i) f is a homeomorphism in the Zariski topology, and

(ii) X is affine if and only if Y is affine; when this occurs and the coordinate ring $R(Y)$ is identified with a subring of $R(X)$, there is a power q of p such that $R(X)^q \subseteq R(Y)$.

Proof. The first property has already been remarked; it is an obvious consequence of the fact that finite morphisms are closed.

To prove (ii), first suppose Y is affine. Then X is affine since finite morphism are affine. On the other hand, when X is affine, a theorem of Chevalley [17, p. 222] says Y is also. The last assertion follows directly from [22, Appendix B, Prop. B. 9].

(2.4) *Remarks.* (a) Any inclusion $A^q \subseteq B \subseteq A$ of coordinate rings, for q a power of p, gives a finite bijective morphism on the associated varieties. This gives a converse to (ii) above, which can be used to show that the condition "there is a finite bijective morphism $X \to Y$" defines an equivalence relation on varieties over k. Cf. [16] for a discussion of the latter in the setting of schemes.

(b) Any f as in (2.3) is a homeomorphism in the étale topology, as well the Zariski topology [16, Théorème 1.1].

(c) Quillen's proof of the stratification theorem in [20] gives some information on the p-power q of (ii) in the case of the morphism $t_{G,E}$ of (2.2): the p-part of the order of the quotient group $N_G(E)/E$, or even $C_G(E)/E$ (improving the argument), is big enough. Similarly, the power q in (2.6) below can be bounded by a product of such terms, taken over all conjugacy classes of elementary subgroups. It would be interesting to have further results in this direction. In the module situation of § 3, the estimate $|C_G(E)/E|_p$ still works for the morphism of (3.2), but for the analogue of (2.6) we have no bound. See (3.4) and the subsequent remark.

Proof of (2.2). Since $V_E^+ = t_{G,E}^{-1} V_{G,E}^+$, the morphism $V_E^+ \to V_{G,E}^+$ is finite and surjective. Thus $V_E/(N_G(E)/C_G(E)) \to V_E^+$ is finite and bijective by (2.1). V_E^+ is affine, since $V_E^+ = (V_E)_e$, where e is the product of the linear functionals defining the subspaces $V_{E,F} = t_{E,F} V_F$ for the subgroups F of index p in E. Clearly, the fixed points in V_E of any $x \in N_G(E) \setminus C_G(E)$ lie in such a $V_{E,F}$, and thus $N_G(E)/C_G(E)$ acts freely on V_E^+. The remaining assertions of (2.2) now follow from (2.1) and part (ii) of (2.3).

It is easy to keep track of the topology in the Quillen stratification, using the fact that $V_{G,E} \cap V_{G,E'} = \bigcup V_{G,F}$, where F ranges over the set of elementary abelian subgroups conjugate to subgroups of both E and E'. (This follows easily from (2.1)). Since there are only finitely many $V_{G,E}$, one easily sees that $V_G \simeq \lim\operatorname{ind} V_{G,E}$, where the inductive limit is taken (in the general sense) with respect to the obvious inclusion morphisms. The next result, which might be called the "glueing theorem," partially lifts this to the varieties V_E, giving quite good information on how to go about building V_G from the V_E's.

(2.5) **Theorem** (*Quillen* [22]). *The natural morphism*

$$\lim\operatorname{ind} V_E \to V_G$$

is bijective and finite. Here the inductive limit is over the category whose objects are the elementary abelian p-subgroups of G, with morphisms all compositions of inclusions and conjugations.

Proof. Each map $V_E \to V_G$ is finite, and there are only finitely many E's. Hence the map $\lim\operatorname{ind} V_E \to V_G$ is finite. The bijectivity follows easily from (2.1), and the proof is complete.

(2.6) **Corollary** (*Quillen* [21]), *Suppose we have an element $\gamma_E \in H_E$ for each elementary abelian p-subgroup E of G, such that each conjugation or restriction map $H_E \to H_F$ carries γ_E to γ_F. Then there is an element $\gamma \in H_G$ and a power q of p such that*

$$\gamma|_E = \gamma_E^q$$

for each elementary abelian p-subgroup E.

Proof. Each such family $\{\gamma_E\}$ defines an element of the coordinate ring of $\lim\operatorname{ind} V_E$. The result now follows from (2.5) and (2.3).

We remark that the same result holds for a family $\{\gamma_E\}$ indexed by any collection $\mathscr{F}$ of elementary p-subgroups closed under conjugation and containing all subgroups of its members. (For one proof, just note that $\lim\operatorname{ind}_{E\in\mathscr{F}} V_E \to \bigcup_{E\in\mathscr{F}} V_{G,E}$ is a bijective finite morphism). Proving this more general result inductively and using [20], one gets the bound on q mentioned in (2.4c).

§ 3. The Main Theorems

As before, G is a finite group and V_G is the variety $\operatorname{Max} H_G$. If M is a finitely generated kG-module, $V_G(M)$ is the cohomology variety defined in the introduction, or equivalently, in [4, 9]. Again we recall from [4] that, if H is a subgroup of G, the transfer map $t_{G,H}\colon V_H \to V_G$ induced by restriction on cohomology rings has the property $t_{G,H} V_H(M) \subseteq V_G(M)$. Our main theorem is a converse to this.

(3.1) **Theorem.** *With the notation above, we have*

$$t_{G,H}^{-1} V_G(M) = V_H(M).$$

Proof. Let $v \in t_{G,H}^{-1} V_G(M)$. Applying (2.2) to H, we can choose an elementary p-subgroup $E \leqq H$ and $s \in V_E^+$ with $t_{H,E}(s) = v$. By [3] or [4], we can choose an elementary p-subgroup E' of G and $s' \in V_{E'}(M)$ with $t_{G,E'}(s') = t_{G,H}(v)$. Finally, again using (2.2), we can choose $E'' \leqq E'$ and $s'' \in V_{E''}^+$ with $t_{E',E''}(s'') = s'$. Then $t_{G,E''}(s'') = t_{G,E'}(s') = t_{G,H}(v) = t_{G,E}(s)$, so (2.1) implies that the pairs (E, s) and (E'', s'') are conjugate by an element of G. By (1.3), $s'' \in V_{E''}(M)$, hence $s \in V_E(M)$, and it follows that $v=t_{H,E}(s)\in V_H(M)$. Thus $t_{G,H}^{-1}V_G(M)\subseteq V_H(M)$, and, as the reverse inclusion has already been noted, the theorem is proved.

Put $V_E^+(M)=V_E^+\cap V_E(M)$ and $V_{G,E}^+(M)=V_{G,E}^+\cap V_G(M)$. Clearly, (3.1) gives $V_E^+(M)=t_{G,E}^{-1}V_{G,E}^+(M)$. Essentially all the results stated in § 2 now carry over for M.

(3.2) **Theorem.** *The variety $V_G(M)$ is the disjoint union of its subvarieties $V_{G,E}^+(M)$, where E ranges over a set of representatives for the conjugacy classes of elementary abelian p-subgroups of G. Moreover, each of the varieties $V_{G,E}^+(M)$ and $V_E^+(M)$ is affine, the group $N_G(E)/C_G(E)$ acts freely on $V_E^+(M)$ and $t_{G,E}$ induces a bijective finite morphism (cf. (2.3))*

$$V_E^+(M)/(N_G(E)/C_G(E))\to V_{G,E}^+(M).$$

Proof. This is clear from (2.1) and the remarks above.

We mention that $V_E^+(M)/(N_G(E)/C_G(E))$ identifies with the image of $V_E^+(M)$ in $V_E^+/(N_G(E)/C_G(E))$, as follows from the fact that the action of $N_G(E)/C_G(E)$ is free. We remark also that $V_E^+(M)$ is empty unless E is contained in a vertex of some indecomposable component of M, by [4] and (3.1).

The next theorem is proved exactly as (2.5).

(3.3) **Theorem.** *Let $\mathscr{F}$ be a family of elementary abelian p-subgroups of G which is closed under conjugation and taking subgroups. Then the natural morphism*

$$\lim_{\substack{\longrightarrow \\ E\in\mathscr{F}}}\text{ind}\, V_E(M)\to V_G(M)$$

is a bijective finite morphism onto a closed subvariety of $V_G(M)$. If $\mathscr{F}$ is the family of all elementary abelian p-subgroups of G, the image is all of $V_G(M)$.

(3.4) **Corollary.** *Let $\mathscr{F}$ be as in (3.3). For any subgroup H of G, let $r_H(M)$ denote the radical ideal in H_H defining $V_H(M)$ as a subvariety of V_H. (If H is a p-group, $r_H(M)$ is the radical of the annihilator of $H^*(H,M)$ in H_H. A similar interpretation can be given in general; see [4]). Suppose for each $E\in\mathscr{F}$, we have an element $\gamma_E\in H_E$ and that for any $E'\in\mathscr{F}$, and any conjugation or restriction map $H_E\to H_{E'}$, the element γ_E is sent to an element of the coset $\gamma_{E'}+r_{E'}(M)$. Then there exists an element $\gamma\in H_G$ and a power q of p such that, for each $E\in\mathscr{F}$,*

$$\gamma|_E\equiv\gamma_E^q \pmod{r_E(M)}.$$

This follows as in the proof of (2.6). We remark that (3.1) says that, for any subgroup F of an elementary abelian p-subgroup E, $r_F(M)$ is the radical of $H_F\cdot \mathrm{res}_{E,F}(r_E(M))$. This can be used, together with the permutation action of $N_G(E)$ on H_E, as a basis for an inductive proof of (3.4) in the spirit of the remark following (2.6). However, this does not lead to a general bound on q here, because we do not know what power of $r_F(M)$ is contained in $H_F\cdot \mathrm{res}_{G,F}(r_G(M))$.

As mentioned earlier, the next result is due to Jon Carlson in the case of elementary ableian p-groups.

(3.5) **Theorem.** *Let M and N be finitely generated kG-modules. Then*

$$V_G(M\otimes_k N)=V_G(M)\cap V_G(N).$$

Proof. Consider $M \otimes_k N$ as module for $G \times G$. Since the irreducible $G \times G$-modules are tensor products of irreducible G-modules, the formulas for cohomology with coefficients in a tensor product yield $V_{G \times G}(M \otimes N) \simeq V_G(M) \times V_G(N)$, compatibly with the usual identification $V_{G \times G} \cong V_G \times V_G$. With this identification, the map $t_{G \times G, G}: V_G \to V_{G \times G} \simeq V_G \times V_G$ arising from the diagonal embedding of G is just the diagonal embedding of the varieties. Hence $t_{G \times G, G}^{-1}(V_G(M)) \times V_G(N))$ is just $V_G(M) \cap V_G(N)$. On the other hand

$$t_{G \times G, G}^{-1}(V_G(M) \times V_G(N)) = t_{G \times G, G}^{-1}(V_{G \times G}(M \otimes N)) = V_G(M \otimes N)$$

by (3.1).

It is interesting to note that there is an obvious parallel result $V_G(M \oplus N) = V_G(M) \cup V_G(N)$, as noted by Carlson [9] in the elementary case.

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